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A Model for Cylindrical Variables with Applications

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Summary

A suitable model when two random variables take values on a cylinder is proposed and its main properties are discussed. The maximum likelihood estimators for the parameters of the model are obtained and a likelihood ratio test of independence is derived. A numerical example is given.

Keywords: CYLINDRICAL VARIABLES; VON MISES DISTRIBUTION; PERIODIC REGRESSION

1. The Model

There are various practical situations where the observations take values on a cylinder (x, θ), $-\infty < x < \infty$, $0 < \theta \leqslant 2\pi$. For some recent examples in rhythmometry, medicine and demography, see Batschelet *et al.* (1973). For examples in biology and climatology, see Bliss (1958).

For most of these situations, we propose the model with the density $f(x, \theta)$ as

$$f(x, \theta) = \{2\pi I_0(k)\}^{-1} \exp\{k \cos(\theta - \mu_0)\} (2\pi\sigma_c^2)^{-\frac{1}{2}} \exp[-\{(x-\mu_c)^2/2\sigma_c^2\}], \tag{1.1}$$

where $-\infty < x < \infty$, $0 < \theta \leqslant 2\pi$, $k > 0$, $0 < \mu_0 \leqslant 2\pi$, $I_0(k)$ is the modified Bessel function of the first kind and order zero and

$$\mu_c = \mu + \sigma k^{\frac{1}{2}}\{\rho_1(\cos\theta - \cos\mu_0) + \rho_2(\sin\theta - \sin\mu_0)\}, \tag{1.2}$$

$$\sigma_c^2 = \sigma^2(1-\rho^2) \quad \text{and} \quad \rho = (\rho_1^2 + \rho_2^2)^{\frac{1}{2}}, \quad 0 \leqslant \rho \leqslant 1. \tag{1.3}$$

The parameters of the model are thus μ, μ_0, k, ρ_1, ρ_2 and σ. The motivation for this model, through a trivariate normal distribution, is left to the Appendix so as not to interrupt the discussion of the main properties of the model.

The form (1.1) implies a significant property of the model in that the conditional distribution of x given θ is $N(\mu_c, \sigma_c^2)$. Consequently

$$E(x \mid \theta) = b_0 + b_1 \cos\theta + b_2 \sin\theta \tag{1.4}$$

and

$$\operatorname{var}(x \mid \theta) = \sigma^2(1-\rho^2), \tag{1.5}$$

where

$$b_0 = \mu - b_1 \cos\mu_0 - b_2 \sin\mu_0, \quad b_1 = \sigma k^{\frac{1}{2}} \rho_1 \quad \text{and} \quad b_2 = \sigma k^{\frac{1}{2}} \rho_2. \tag{1.6}$$

Of course the regression of x on θ defined by (1.4) is often fitted to periodic data.

The marginal distribution of θ is von Mises with mean direction μ_0 and concentration parameter k, i.e. $M(\mu_0, k)$, but the marginal distribution of x is complicated.

For $\rho = 0$, x and θ are independently distributed as $N(\mu, \sigma^2)$ and $M(\mu_0, k)$ respectively. Also $\rho = 1$ implies "perfect correlation" between x and θ, since in this case

$$x = b_0 + b_1 \cos\theta + b_2 \sin\theta$$

with probability one. Note that the quantity ρ must always be positive in view of the continuity of $E(x \mid \theta)$ and the periodicity of θ. From the above it can be seen that the parameter ρ can be described as a parameter of dependence between x and θ.

2. THE MAXIMUM LIKELIHOOD ESTIMATORS

Let (x_i, θ_i), $i = 1, \ldots, n$, be a random sample from (1.1) and let $\bar{R}$ and $\bar{x}_0$ be respectively the mean resultant length and the mean direction of the θ_i's, i.e.

$$\bar{R}\cos\bar{x}_0 = n^{-1}\sum_{i=1}^{n}\cos\theta_i \quad \text{and} \quad \bar{R}\sin\bar{x}_0 = n^{-1}\sum_{i=1}^{n}\sin\theta_i. \tag{2.1}$$

Write

$$\left.\begin{aligned} &x_{1i} = x_i, \quad x_{2i} = \cos\theta_i, \quad x_{3i} = \sin\theta_i, \quad i = 1, \ldots, n \\ &s_j^2 = \operatorname{var}(x_j), \quad r_{jk} = \operatorname{corr}(x_j, x_k), \quad j \neq k, \quad j = 1, 2, 3. \end{aligned}\right\} \tag{2.2}$$

and

Initially, let us consider the six parameters of the model as μ_0, k, b_0, b_1, b_2 and σ_c. The log of the likelihood function is then given by

$$\text{const} + \left(-n\log_e I_0(k) + k\sum_{i=1}^{n}\cos(\theta_i - \mu_0)\right) - \left(n\log_e\sigma_c + (2\sigma_c^2)^{-1}\sum_{i=1}^{n}(x_{1i} - b_0 - b_1 x_{2i} - b_2 x_{3i})^2\right). \tag{2.3}$$

It follows that the maximum likelihood estimators of the parameters are given by

$$\hat{\mu}_0 = \bar{x}_0, \quad \hat{k} = A^{-1}(\bar{R}), \tag{2.4}$$

$$\hat{b}_1 = \frac{s_1}{s_2}\,\frac{r_{23}r_{13} - r_{12}}{r_{23}^2 - 1}, \quad \hat{b}_2 = \frac{s_1}{s_3}\,\frac{r_{12}r_{23} - r_{13}}{r_{23}^2 - 1}, \tag{2.5}$$

$$\hat{b}_0 = \bar{x} - \hat{b}_1\bar{R}\cos\bar{x}_0 - \hat{b}_2\bar{R}\sin\bar{x}_0 \tag{2.6}$$

and

$$\hat{\sigma}_c^2 = s_1^2(1 - r_{1.23}^2), \tag{2.7}$$

where $r_{1.23}$ is the multiple correlation coefficient of x_1 with x_2 and x_3

$$r_{1.23}^2 = (r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23})/(1 - r_{23}^2). \tag{2.8}$$

Further, $A(k) = I_1(k)/I_0(k)$, where $I_1(k)$ is the modified Bessel function of the first kind of order one. Note that (2.4) follows from (2.3) on using the maximum likelihood estimators for the von Mises distribution (see, for example, Mardia, 1972, p. 122), whereas (2.5)–(2.7) follow on using the usual normal theory technique.

On substituting the expressions for the above estimators in (1.2) and (1.3) it is found that the maximum likelihood estimators of the original parameters μ, μ_0, k, ρ_1, ρ_2 and σ are given by

$$\hat{\mu} = \bar{x} + (1 - \bar{R})(\hat{b}_1\cos\bar{x}_0 + \hat{b}_2\sin\bar{x}_0), \tag{2.9}$$

$$\hat{\sigma}^2 = s_1^2(1 - r_{1.23}^2) + \{(\hat{b}_1^2 + \hat{b}_2^2)/\hat{k}\}, \tag{2.10}$$

$$\hat{\rho}_1 = \hat{b}_1/(\hat{\sigma}\hat{k}^{\frac{1}{2}}), \quad \hat{\rho}_2 = \hat{b}_2/(\hat{\sigma}\hat{k}^{\frac{1}{2}}). \tag{2.11}$$

3. A TEST OF INDEPENDENCE

Since ρ is a measure of dependence between x and θ, we wish to test

$$H_0\colon \rho = 0 \quad \text{against} \quad H_1\colon \rho \neq 0.$$

If L_i is the likelihood under H_i, $i = 1, 2$, then it is seen from Section 2 that the likelihood ratio λ is given by

$$\lambda = \max L_0/\max L_1 = (1 - r_{1.23}^2)^{n/2}.$$

Hence for large n and under H_0 we have

$$-2\log_e \lambda = -n \log_e (1-r_{1.23}^2) \simeq \chi_1^2, \tag{3.1}$$

since $\rho = 0$ if and only if $\rho_1 = \rho_2 = 0$.

It should be noted that the test of independence depends only on the multiple correlation coefficient $r_{1.23}^2$. In fact Mardia (1976) has indeed proposed this quantity as a measure of dependence on intuitive grounds. The proposed model not only puts the measure on a firm footing, but also provides its asymptotic distribution.

Let $\rho_{1.23}$ be the population multiple correlation coefficient. We now examine the relationship between $\rho_{1.23}$ and ρ. Following the same argument as to derive (2.7) we find that

$$\sigma_c^2 = V(1-\rho_{1.23}^2), \tag{3.2}$$

where $V = \text{var}(x)$. Using (1.3), we thus have

$$\sigma^2(1-\rho^2) = V(1-\rho_{1.23}^2). \tag{3.3}$$

Now using $\text{var}(x|\theta)$ and $E(x|\theta)$ given by (1.4) and (1.5) in

$$V = E\{\text{var}(x|\theta)\} + \text{var}\{E(x|\theta)\},$$

it is found that

$$V = \sigma^2(1-\rho^2) + \text{var}(b_1 \cos\theta + b_2 \sin\theta). \tag{3.4}$$

Also on using the results

$$\text{var}(\cos\theta) = a_1 \sin^2\mu_0 + a_2\cos^2\mu_0, \quad \text{var}(\sin\theta) = a_1\cos^2\mu_0 + a_2\sin^2\mu_0$$
$$\text{and} \quad \text{cov}(\cos\theta, \sin\theta) = (a_2 - a_1)\cos\mu_0 \sin\mu_0,$$

where $a_1 = A(k)/k$ and $a_2 = A'(k) = 1 - a_1 - A^2(k)$, it is found that (3.4) becomes

$$V = \sigma^2(1-\rho^2) + \sigma^2 k\rho^2 U, \tag{3.5}$$

where $U = a_1\sin^2(\mu_0 - \varepsilon) + a_2\cos^2(\mu_0-\varepsilon)$ and ε is defined by $\rho_1 = \rho\cos\varepsilon$, $\rho_2 = \rho\sin\varepsilon$.

Finally, on substituting (3.5) in (3.3), it is found that

$$\rho_{1.23}^2 = \left(1 + \frac{(1-\rho^2)}{k\rho^2 U}\right)^{-1}. \tag{3.6}$$

It can be seen that $\rho^2 = 0$ implies $\rho_{1.23}^2 = 0$ and that $\rho^2 = 1$ implies $\rho_{1.23}^2 = 1$. Note that U does not depend on ρ as long as we take $\rho > 0$. Hence $\partial\rho_{1.23}^2/\partial\rho^2 = kU/\{1+\rho^2(kU-1)\}^2, k>0$, $U>0$, so that $\rho_{1.23}^2$ is a monotonically increasing function of ρ^2. However, $r_{1.23}^2$ is not a function of ρ^2 only, and therefore, in general, a test of independence based on ρ^2 will be different from the above test.

4. A NUMERICAL STUDY

Table 1 gives 28 observations consisting of the January surface wind direction (θ) and temperature (x) at 12 h GMT at Kew for the years 1956–60. For meteorological reasons, the observations were separated by a three-day interval and had wind speed greater than 10 knots. It is found that

$$\bar{x}_0 = 231°, \quad \bar{R} = 0{\cdot}493, \quad \hat{k} = 1{\cdot}14.$$

Further,

$$\hat{b}_1 = -4{\cdot}67, \quad \hat{b}_2 = -1{\cdot}80, \quad \hat{\rho} = 0{\cdot}694, \quad \hat{\sigma}^2 = 45{\cdot}51.$$

Using $\bar{x} = 44{\cdot}21$, it is found that $\hat{\mu} = 46{\cdot}405$. The fitted regression of x on θ is

$$x = 42{\cdot}1 + 5{\cdot}0\cos(\theta - 201°)$$

with the residual variance $\hat{\sigma}_c^2 = 23{\cdot}57$.

TABLE 1

January surface wind direction θ (in degrees) and temperature x (in Fahrenheit) at 12 h GMT at Kew during 1956–60

θ	x	θ	x	θ	x
250	52	210	48	320	37
210	41	190	52	320	33
250	41	280	43	190	47
90	31	180	46	210	51
210	53	260	38	70	42
210	47	150	40	260	53
350	43	170	49	240	46
340	43	230	48	200	51
30	41	240	37	270	39
210	46				

Fig. 1 shows the fitted curve and the observed points. In fact, we have

$$r_{1.23}^2 = 0{\cdot}3216, \quad -n\log_e(1-r_{1.23}^2) = 10{\cdot}8.$$

Using the approximation (3.1) we reject the hypothesis of independence at the 1 per cent level of significance; the 1 per cent value of χ_2^2 being 9·21. Even though $n = 28$ is not very large, the order of magnitude of $n\log_e(1-r_{1.23}^2)$ suggests strongly that the conclusion is valid.

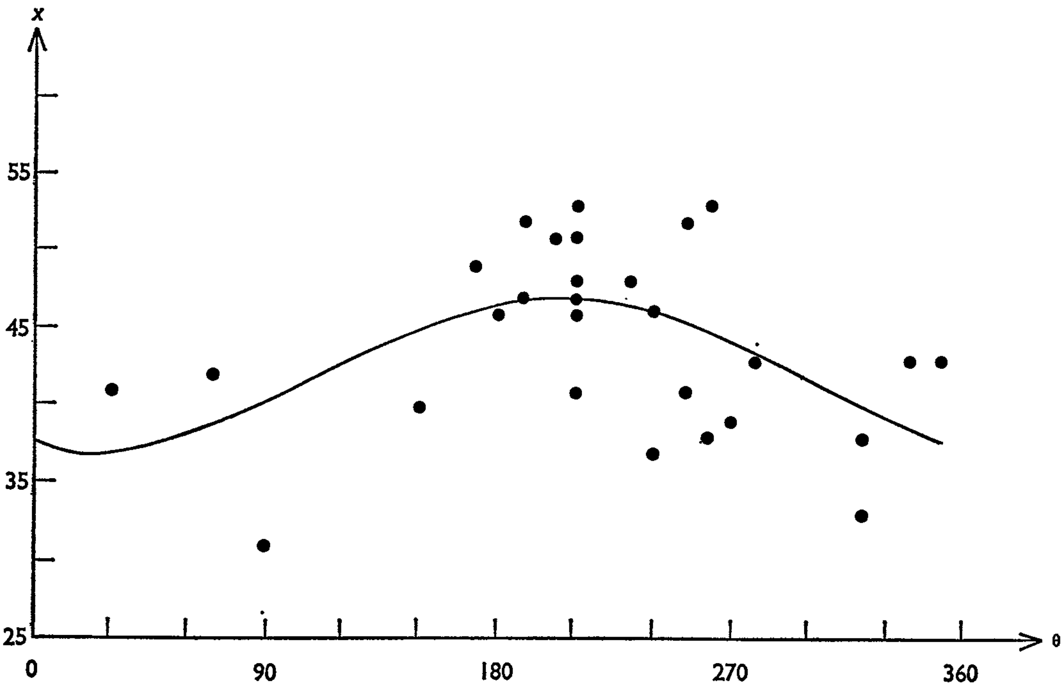

FIG. 1. The observed points and fitted curve for the Kew data in Table 1.

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APPENDIX

It is natural to look for a model which has the distributions of x and θ as $N(\mu, \sigma^2)$ and $M(\mu_0, k)$ respectively when the variables are independent. Further, a suitable model should have built in dependence between x and θ in a simple way. We recall that Fisher constructed $M(\mu_0, k)$ as the conditional distribution of $N_2(\boldsymbol{\mu}, k^{-1}\boldsymbol{\Sigma})$ on the unit circle where

$$\boldsymbol{\mu}' = (\cos\mu_0, \sin\mu_0),\; k>0.$$

Hence if $x_1 = x$, $x_2 = r\cos\theta$, $x_3 = r\sin\theta$ where (x_1, x_2, x_3) is distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then the distribution of (x_1, x_2, x_3) given $r = 1$ can be looked upon as a suitable distribution for (x, θ). The above discussion implies restricting ourselves to the following choice of the parameters;

$$\mu_1 = \mu,\quad \mu_2 = \cos\mu_0,\quad \mu_3 = \sin\mu_0,\quad \sigma_{11} = \sigma^2,\quad \sigma_{22} = \sigma_{33} = 1/k,\quad \sigma_{23} = 0,\quad k>0.$$

Let us write $\rho_1 = \operatorname{corr}(x_1, x_2)$, $\rho_2 = \operatorname{corr}(x_1, x_3)$. With these values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, the p.d.f. of (x, θ) is found to be (1.1). Note that for (a) $\rho = 0$, x and θ are independent $N(\mu, k)$ and $M(\mu_0, k)$ and (b) as $\rho \to 1$, we have $x = a + b_1\cos\theta + b_2\sin\theta$ with probability one. Hence ρ can be regarded as a parameter of dependence.